

B. R. Belostotskii

Inzhenerno-Fizicheskii Zhurnal, Vol. 15, No. 2, pp. 219-227, 1968

UDC 621.375.9

A method is presented for the calculation of the temperature field of continuous-wave lasers, given constant thermophysical characteristics for the active medium and that these are dependent on temperature.

We know that a portion of the pumping energy in a laser is unavoidably dissipated as heat, resulting in the heating of the working substance. The rise in the temperature of the working substance leads to a pronounced impairment of the laser characteristics [1-4, etc.], and in a number of cases it leads to the appearance of substantial thermal stresses in the working medium of the laser. The problem of cooling is therefore basic to the design of laser equipment.

Let us take the physical model of the process from [5], since the processes of heat transfer exhibit considerably greater inertia than the nonradiation-transition processes which cause the heating of the working substance; the latter is treated as a specimen with internal heat sources distributed in a specific manner throughout the entire volume. For a working substance in the form of a circular cylinder of radius R under identical conditions for the cooling of each element of the side surface and the thermal insulation of the ends, the problem with respect to the temperature field is formulated by the following system of differential equations:

$$I \left\{ \begin{aligned} & c_{p_1} \gamma_1 \frac{\partial \theta(r_1, Fo)}{\partial Fo} \\ & = \operatorname{div} \lambda_1 \operatorname{grad} \theta(r_1, Fo) + Ki(r_1), \quad (1) \\ & Fo = 0, \quad \theta(r_1, 0) = 0, \quad (2) \\ & -\lambda_1 \left[\frac{\partial \theta(r_1, Fo)}{\partial r_1} \right]_{r_1=1} \\ & = Bi [\theta(r_1, Fo)]_{r_1=1}, \quad (3) \\ & c_{p_1} = \frac{c_p(\theta)}{c_{p_f}} = c_p(\theta), \\ & \lambda_1 = \frac{\lambda(\theta)}{\lambda_f} = \lambda_1(\theta), \quad (4) \\ & \gamma_1 = \frac{\gamma(\theta)}{\gamma_f} = \gamma_1(\theta). \end{aligned} \right.$$

§1. If the thermophysical characteristics are independent of temperature, the solution of system I, given the uniform distribution of the heat sources ($Ki = \text{const}$), is known [6]:

$$\theta(r_1, Fo) = \frac{Ki}{4} \left[\left(1 + \frac{2}{Bi} - r_1^2 \right) - \sum_{n=1}^{\infty} \frac{4}{\mu_n^2} A_n J_0(\mu_n r_1) \exp(-\mu_n^2 Fo) \right]. \quad (5)$$

As $Fo \rightarrow \infty$, (5) changes into a relation which characterizes the temperature distribution in the steady-state regime:

$$\theta(r_1) = \frac{Ki}{4} \left(1 + \frac{2}{Bi} - r_1^2 \right). \quad (6)$$

Bearing in mind that the maximum temperature of the process, as a rule, is specified in the calculation, and that with uniform pumping the maximum temperature is developed at the center of the specimen, it makes sense to present the dimensionless temperature in the form

$$\theta(r_1) = \frac{T(r_1) - T_f}{T(0) - T_f}. \quad (7)$$

Then (6) is written as

$$\theta(r_1) = 1 - \frac{r_1^2}{1 + \frac{2}{Bi}}. \quad (8)$$

From (6) we draw the obvious conclusion that the profile of the temperature field and, consequently, the temperature difference $\Delta\theta_{i,k}$ between any two points of the specimen, are independent of the cooling conditions and are functions exclusively of the Kirpichev number Ki . Indeed,

$$\Delta\theta_{i,k} = \frac{Ki}{4} (r_{1,i}^2 - r_{1,k}^2). \quad (9)$$

Consequently, it is enough to know the temperature of a single point of the specimen to construct the cross-sectional temperature field.

We can judge the relative nonuniformity of the temperature field from the magnitude of the dimensionless temperature $\theta(r_1)$, which was introduced in accordance with (7). It is interesting to note that the quantity $\theta(r_1)$ is independent of the pumping level and is not associated with a specific specimen. This enables us to speak of the thermal regime for lasers in general. The relationship between the dimensionless temperature and the efficiency of cooling is shown in Fig. 1. The Biot number in specific laser circuits may vary over a wide range. Thus, under conditions of water cooling for ruby specimens, the Biot number, as a rule, does not exceed 2, while in the case of glass it may go as high as 400. As follows from Fig. 1, for ruby rods when $Bi = 2$ the possibilities of dropping the temperature level have by no means been exhausted, and on the basis of these considerations it is advisable in a number of cases to resort to cooling by means of a flow of low-boiling liquids (nitrogen, oxygen, Freon). For glass materials, the available approaches under conditions of water cooling when $Bi > 20$ make no sense

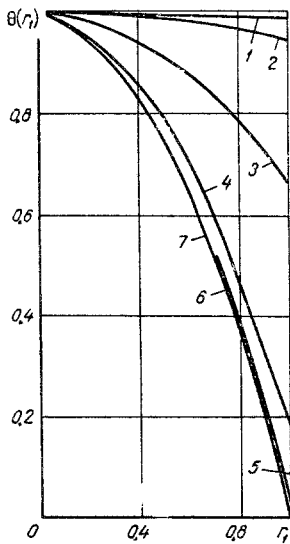


Fig. 1. Effect of efficiency of cooling system on thermal regime of working substance: 1) $Bi = 0.01$; 2) 0.1; 3) 1; 4) 10; 5) 20; 6) 100; 7) ∞ .

($\theta(1) < 0.1$). This last statement is extremely important, since it excludes—for glass materials—the possibility of constructing cooling systems designed for large liquid flow rates, high pressures, elevated requirements imposed on hose strength, on packing glands, etc., and it also excludes the possibility of turning to cooling systems based on low-boiling liquids.

When the maximum temperature difference ΔT_{\max} between the center and the surface of the working substance is specified, we draw analogous conclusions on examination of the relationship which follows from (6):

$$\theta' = \frac{2}{Bi}, \quad (10)$$

where

$$\theta' = \frac{T(1) - T_l}{\Delta T_{\max}}. \quad (11)$$

The maximum temperature and the maximum difference in temperatures between the center and the surface of the laser rod can be determined if we know the temperature at which the laser operation is still possible. Stricter limitations on the magnitude of ΔT_{\max} will ensue from the requirement for greater efficiency, greater monochromaticity, lower threshold pumping power, preservation of optical uniformity associated with the temperature gradient, and the requirements with respect to the thermal stability of the material, etc.

Thus, the calculation of the thermal regime for a continuous-wave laser involves analysis of (6)–(11). One of these may be a test equation—for example, formula (9)—to coordinate the requirements with respect to material thermal stability.

§2. The pumping-energy absorption density through the volume of the substance is not generally constant. However, in the thermal calculations this fact, as a rule, is not considered. Introduction of correction factors for the assumption of uniformity frequently markedly alters the results derived on the basis of the formulas cited above. We will present the field for the distribution of the density $q(r_1)$ of the absorbed pumping energy by means of a polynomial in even powers of r_1

$$q(r_1) = \sum_{i=1}^l (-1)^{i+1} a_i r_1^{2(i-1)} \quad (12)$$

or in dimensionless form

$$Ki(r_1) = \sum_{i=1}^l (-1)^{i+1} Ki_i r_1^{2(i-1)}. \quad (13)$$

The problem in this case is formulated by a system of equations analogous to system I. We will obtain the solution of this system for $Ki = Ki(r_1)$ in a form similar to (5). For this we will present the general solution of the problem as the sum of the solutions for two special problems: the steady-state regime in the case of nonuniform pumping $V(r_1)$ and the non-steady-state regime when there are no heat sources $U(r_1, Fo)$. It is not difficult to derive the analytical expression $V(r_1)$ in the form

$$V(r_1) = \sum_{i=1}^l (-1)^{i+1} \frac{Ki_i}{4i^2} \left(1 + \frac{2i}{Bi} - r_1^{2i} \right). \quad (14)$$

The solution for $U(r_1, Fo)$ is known [6]:

$$U(r_1, Fo) = \sum_{i=1}^{\infty} \frac{2J_0(\mu_n r_1) \exp(-\mu_n^2 Fo)}{\mu_n [J_0^2(\mu_n) + J_1^2(\mu_n)]} \times \int_0^1 r_1 V(r_1) J_0(\mu_n r_1) dr_1. \quad (15)$$

The general solution for the problem is obtained by substituting (14) into (15):

$$\theta(r_1, Fo) = \sum_{i=1}^l (-1)^{i+1} \frac{Ki_i}{4i^2} \left(1 + \frac{2i}{Bi} - r_1^{2i} \right) + \sum_{i=1}^l \sum_{n=1}^{\infty} (-1)^{i+1} \frac{J_0(\mu_n r_1) \exp(-\mu_n^2 Fo)}{2\mu_n [J_0^2(\mu_n) + J_1^2(\mu_n)]} \times \int_0^1 r_1 \frac{Ki_i}{i^2} \left(1 + \frac{2i}{Bi} - r_1^{2i} \right) J_0(\mu_n r_1) dr_1. \quad (16)$$

Assuming $i = 1$, we change the last expression into (5). When $i = 2$, we have

$$\theta(r_1, Fo) = \frac{1}{4} Ki_1 \left(1 + \frac{2}{Bi} - r_1^2 \right) - \frac{1}{16} Ki_2 \left(1 + \frac{4}{Bi} - r_1^4 \right) - \sum_{n=1}^{\infty} \frac{Ki_1}{\mu_n^2} A_n J_0(\mu_n r_1) \exp(-\mu_n^2 Fo) + \sum_{n=1}^{\infty} \frac{Ki_2}{\mu_n^2} A_n J_0(\mu_n r_1) \times \left(1 + \frac{2}{Bi} - \frac{4}{\mu_n^2} \right) \exp(-\mu_n^2 Fo). \quad (17)$$

Having determined the gradient for the temperature field (16) in the steady-state regime, as in the case of uniform pumping, we prove to our satisfaction that the profile for the temperature field—given nonuniform density of pumping-energy absorption—is independent of the efficiency of the method by means of which the heat is removed. On the basis of temperature measurements at a single point of the specimen, this fact

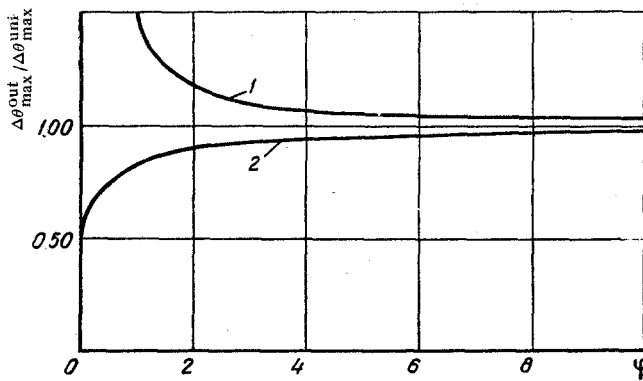


Fig. 2. Effect of nonuniformity of pumping energy absorption field on the value of temperature drop between the center and the surface of the working substance (fractions $\Delta\theta_{\max}^{\text{uni}}$): 1) for $a_2 > 0$; 2) for $a_2 < 0$.

also enables us to reproduce the entire temperature profile.

We will present the maximum temperature difference (in dimensionless form), i.e., the temperature difference $\Delta\theta_{\max}^{\text{out}}$ between the center of the specimen and its surface, in the form of the following sum:

$$\Delta\theta_{\max}^{\text{out}} = \Delta\theta_{\max}^{\text{uni}} + \zeta, \quad (18)$$

where $\Delta\theta_{\max}^{\text{uni}} = \bar{K}i/4$ is the maximum temperature difference for uniform heat release;

$$\bar{K}i = \sum_{i=1}^i (-1)^{i+1} \frac{Ki_i}{i}$$

is the Kirpichev number which corresponds to the volume-averaged density of heat sources; ζ is the correction factor to account for the nonuniformity in the distribution of the pumping through the volume of the specimen, and this factor is determined from the relationship

$$\zeta = \sum_{i=1}^i (-1)^i \frac{Ki_i}{4i} \left(1 - \frac{1}{i} \right). \quad (19)$$

From (18) we can calculate the magnitude of the maximum temperature gradient developed in the active specimen and we can determine the magnitude of the correction factor ζ which should be introduced into the calculation to account for the nonuniformity in the field of pumping-energy absorption.

Figure 2 shows the temperature difference between the center and the surface for the case $i = 2$ in fractions

of $\Delta\theta_{\max}^{\text{uni}}$ for various degrees of nonuniformity in the field of pumping-energy absorption. We can judge the nonuniformity of the field of pumping-energy absorption from the fact that $\varphi = a_1/a_2$. When $a_2 > 0$, the pumping-energy density increases toward the center. The index of temperature-field nonuniformity lies in the range $(1; \infty)$: $a_1/a_2 = \infty$ corresponds to a uniform pumping-energy absorption density; $a_1/a_2 = 1$ is the maximum possible nonuniformity (the surface of substance does not absorb the pumping). As follows from Fig. 2 (curve 1), the difference in temperatures between the center and the surface for $a_1/a_2 = 1$ exceeds the corresponding temperature difference for uniform pumping by a factor of 1.5.

When $a_2 < 0$, the density of the absorbed pumping energy diminishes toward the center. The index of nonuniformity for the temperature field may vary in the range $(0; \infty)$: $a_1/a_2 = \infty$ is a uniform absorption field; $a_1/a_2 = 0$ is the maximum nonuniformity (the pumping energy does not reach the center). When $a_1/a_2 = 0$, the magnitude of the correction factor ζ attains its maximum value and is equal to 0.5, but its sign is opposite to that of the case in which $a_2 > 0$ (Fig. 2, curve 2).

Thus, when $a_2 > 0$ the nonuniformity is aggravated, and when $a_2 < 0$ the nonuniformity of the temperature field is reduced in the steady-state regime. The variations in the temperature difference between the center and the specimen surface lies in the range $\pm 1.5\Delta\theta_{\max}^{\text{uni}}$. In the laser variants which are possible in practical terms, the nonuniformity $\varphi > 5$, so that we can take a uniform field of pumping-energy absorption as the initial field (with an accuracy of up to 5%) in calculations for the steady-state thermal regime of the specimen.

The surface temperature for the working substance is determined from (15):

$$\theta(1) = \frac{\bar{K}i}{2Bi}. \quad (20)$$

As we can see, the surface temperature of the body is independent of the degree of nonuniformity for the field of absorbed pumping radiation, which should be in accord with the definition of the steady-state regime.

The effect of absorption nonuniformity on the maximum temperature developed at the rod axis can be traced in the equation

$$\frac{\theta^{\text{out}}(0)}{\theta^{\text{uni}}(0)} = 1 - \frac{\sum_{i=1}^i (-1)^{i+1} \left(1 - \frac{1}{i} \right) \frac{Ki_i}{i}}{\left(1 + \frac{2}{Bi} \right) \bar{K}i}. \quad (21)$$

Table 1

Effect of the Nonuniformity of the Field of Pumping-energy a_1/a_2 on the Ratio $\theta^{\text{out}}(0)/\theta^{\text{uni}}(0)$ for Various Values of Bi ($i = 2, a_2 > 0$)

Bi	a_1/a_2					
	1	1.25	2	5	10	15
0.01	1.002	1.001	1.001	1.000	1.000	1.000
0.1	1.024	1.016	1.008	1.003	1.001	1.001
1.0	1.167	1.111	1.055	1.018	1.009	1.006
10.0	1.417	1.277	1.138	1.046	1.022	1.014
100.0	1.490	1.326	1.163	1.054	1.026	1.017
∞	1.500	1.333	1.167	1.056	1.026	1.017

Table 2

Effect of Nonuniformity in the Field of Pumping-energy Absorption a_1/a_2 on the Ratio $\theta^{\text{out}}(0)/\theta^{\text{uni}}(0)$ for Various Values of Bi ($i = 2, a_2 < 0$)

Bi	a_1/a_2									
	0	0.1	0.25	0.5	1	2	5	10	15	
0.01	0.997	0.998	0.998	0.999	0.999	1.000	1.000	1.000	1.000	
0.1	0.976	0.980	0.984	0.988	0.992	0.995	0.998	0.999	0.999	
1.0	0.833	0.861	0.888	0.917	0.944	0.967	0.985	0.992	0.995	
10.0	0.583	0.653	0.722	0.792	0.861	0.917	0.962	0.980	0.987	
100.0	0.510	0.592	0.673	0.755	0.837	0.902	0.955	0.977	0.984	
∞	0.500	0.583	0.667	0.750	0.833	0.9	0.955	0.976	0.984	

In Table 1 ($a_2 > 0$) and in Table 2 ($a_2 < 0$) for the case $i = 2$ we present the results from a numerical evaluation of the ratio $\theta^{\text{out}}(0)/\theta^{\text{uni}}(0)$ for various values of the nonuniformity φ and for various Biot numbers.

It is natural that the results shown graphically in Fig. 2 must be treated as limit values for the ratio $\theta^{\text{out}}(0)/\theta^{\text{uni}}(0)$, i. e., as $\text{Bi} \rightarrow \infty$.

As follows from the tables, the temperature at the center may vary substantially as a function of pumping-field profile. However, under actual conditions, when $\varphi > 5$, we are quite justified in assuming for the calculations that the pumping field is uniform. Thus, for a ruby when $\text{Bi} \leq 2, a_2 > 0$, and $i = 2$, the error of such an approximation is less than 3.7%.

§3. The foregoing is in need of refinement for the case in which the thermophysical characteristics of the working substance are functions of temperature. The difficulty of dealing with this problem is associated with the nonlinearity of system I.

The differential heat-conduction equation for the steady-state regime can be linearized for the new variable θ_f , which is introduced on the basis of the following relationship:

$$\theta_f = \frac{1}{\bar{\lambda}_1} \int_0^\theta \lambda_1(\theta) d\theta, \quad \bar{\lambda}_1 = \int_0^1 \lambda_1(\theta) d\theta. \quad (22)$$

Further, bearing in mind that

$$\lambda_1(\theta) = 1 + k_{\lambda_1} \theta + k_{\lambda_2} \theta^2 + \dots \quad (23)$$

can be replaced by the approximation

$$\lambda_1(\theta) = 1 + k_{\lambda} \theta, \quad (24)$$

where the coefficient k_{λ} is found from the variation of (24) for the standard deviation, the relationship between θ and θ_f is easily established in the form

$$\theta = \frac{1}{k_{\lambda}} (\sqrt{1 + 2k_{\lambda} \bar{\lambda}_1 \theta_f} - 1). \quad (25)$$

The nonlinearity of the boundary condition of the third kind is eliminated by the introduction of the coefficient ε according to the relation

$$\varepsilon = \frac{1}{1 + \frac{k_{\lambda}}{2} \theta(1)} \quad (26)$$

and its subsequent replacement by the approximate value

$$\bar{\varepsilon} = 1 - \frac{k_{\lambda}}{4}. \quad (27)$$

Transformed system I for the steady-state regime has the form

$$\text{II} \left\{ \begin{aligned} & \frac{1}{r_1} \frac{d}{dr_1} \left[r_1 \frac{d\theta_f(r_1)}{dr_1} \right] + \text{Ki}_f(r_1) = 0, \quad (28) \\ & - \left[\frac{d\theta_f(r_1)}{dr_1} \right]_{r_1=1} = \text{Bi} \varepsilon [\theta_f(r_1)]_{r_1=1}, \quad (29) \\ & \left[\frac{d\theta_f(r_1)}{dr_1} \right]_{r_1=0} = 0. \quad (30) \end{aligned} \right.$$

The solution of system II is known and has a form analogous to (14). With consideration of (25), we find our answer as

$$\theta(r_1) = \frac{1}{k_{\lambda}} \times \left(\sqrt{1 + k_{\lambda} \sum_{i=1}^i (-1)^{i+1} \frac{\text{Ki}_i}{2i^2} \left(1 + \frac{2i}{\varepsilon \text{Bi}} - r_1^{2i} \right)} - 1 \right). \quad (31)$$

If we expand the radicand in series and limit ourselves to the first two terms, (31) changes to the following:

$$\theta(r_1) = \sum_{i=1}^i (-1)^{i+1} \frac{\text{Ki}_i}{4i^2} \left(1 + \frac{2i}{\varepsilon \text{Bi}} - r_1^{2i} \right). \quad (32)$$

Assuming in the latter that $\varepsilon = 1$, we derive formula (14).

As follows from (32), we can assume in approximate terms as before that the temperature profile is independent of the cooling conditions, and all of our considerations with regard to the effect of nonuniform density of absorbed pumping energy remain valid.

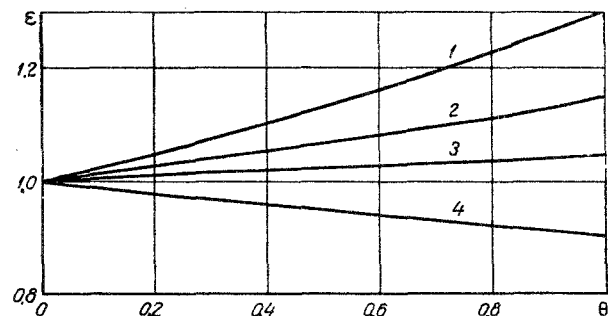


Fig. 3. Coefficient ε versus temperature for a number of active materials: 1) ruby, temperature range 200–300° K; 2) ruby, 300–400; 3) CaWO_4 , 200–300; 4) neodymium glass, 200–300.

Having approximated the results from the study of the thermal conductivity as a function of temperature in the range 200–300° K for such active substances as neodymium glass [7], ruby, and calcium tungstate [8], activated by Nd³⁺ ions by the above-described method, for the ruby we find $k_\lambda = -0.46$, for the calcium tungstate we find $k_\lambda = -0.083$, and for the glass we find $k_\lambda = 0.22$. The error introduced into the calculation in the derivation of (27), and the advisability of correcting the results through consideration of the relationship between the thermophysical characteristics of the active substance and the temperature both follow from the graph shown in Fig. 3. The averaging of ε for the calcium tungstate is quite permissible. In the case of the ruby, such averaging may yield perceptible error. However, under actual conditions the ruby rods operate under conditions only slightly hotter than the cooling medium, and it is therefore not advisable to consider the relationship between the thermophysical characteristics and the temperature. For example, if the overheating does not exceed 20°, in the temperature range 300–400° K (see Fig. 3, curve 2, $\theta = 0.2$) for the ruby the averaging error with respect to ε is less than 2.5%.

Since in the temperature region above 20° K, for the working substances with the exception of glass, the quantity k_λ is negative, and this causes the results of the thermal-regime calculations for the specimens to be slightly exaggerated. It is therefore necessary to take into consideration the variability in the coefficient of thermal conductivity only for operating temperatures below 100° K in the case of the ruby and calcium tungstate, since it is here that the thermal conductivity exhibits a sharp rise. For neodymium glasses, with temperatures substantially in excess of the cooling medium, we also require refinement. The results can be corrected by successive approximation with formula (32). For very large $Bi > 100$, i. e., when the surface temperature is virtually coincident with the temperature of the cooling medium (boundary conditions of the I-st kind), (32) changes into the exact solution of the nonlinear system I:

$$\theta(r_1) = \frac{1}{k_\lambda} \left(\sqrt{1 + k_\lambda \sum_{i=1}^i (-1)^{i+1} \frac{Ki_i}{2^i} (1 - r_1^{2^i})} - 1 \right). \quad (33)$$

§4. The above-derived conclusions can be used in certain cases for the development of pulse lasers.

Here the basic calculational relationship (6) should be used in the form

$$\theta(r_1) = \frac{\bar{\theta}_{pul}}{4Fo_c} \left(1 + \frac{2}{Bi} - r_1^2 \right), \quad (34)$$

where $\bar{\theta}_{pul}$ is the change in the volume-averaged temperature of the working substance during the pumping period.

Results from the comparison of relationship (34) with the formulas for the calculation of the temperature field of the working substance in the pulse laser for the end of the cycle $Fo = Fo_c$ show [5] that the relative error in the calculations of the temperatures for the center of the working substance does not exceed 10% when $Fo_c < 0.5$ for values of $Bi \leq 100$. In comparison with the center of the working substance, the relative error in the determination of its surface temperature is somewhat higher. As Fo_c and Bi diminish, the errors also become smaller.

NOTATION

τ_c is the time between pumping pulses; Bi is the Biot number; Fo is the Fourier number; Ki is the Kirpichev number. Other notations are taken from [6].

REFERENCES

1. J. P. Wittke, Appl. Phys., 33, 2333, 1962.
2. A. A. Kaminskii, L. S. Kornienko, and A. M. Prokhorov, ZhETF, 48, no. 5, 1965.
3. G. Burns and N. Marshall, J. Appl. Phys., 34, no. 3, 703, 1963.
4. Yu. A. Anan'ev, N. A. Kozlov, A. A. Mak, and A. I. Stepanov, ZhPS [Journal of Applied Spectroscopy], 5, no. 1, 1966.
5. B. R. Belostotskii, Trudy KuAI, no. 24, 211, 1967.
6. A. V. Luikov, Theory of Heat Conduction [in Russian], Gostekhizdat, 1952.
7. E. H. Ratclif, Glass Technology, 4, no. 4, 1963.
8. M. G. Holland, J. Appl. Phys., 33, no. 9, 1962.

27 December 1967

Opticomechanical Combine,
Leningrad